An Alternative Proof of Plantholt's Theorem on Edge-colouring Using a Recent Recolouring Procedure

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A k-edge-colouring of a (simple) graph G is an assignment of k colours to the edges of G so that no two adjacent edges are coloured the same. The chromatic index of G, denoted $\chi'(G)$, is the least k for which G admits a k-edge-colouring. By Vizing's Theorem (1964), $\chi'(G)$ is either the maximum degree Δ of G, or it is $\Delta + 1$, in which case G is said to be *Class 1* and *Class 2*, respectively. Although deciding if a given general graph G is Class 1 is an NP-complete problem (Holyer, 1981), Plantholt's Theorem states that an n-vertex graph G with a universal vertex is Class 2 if and only if it is *overfull*, i.e. has at most $\Delta\lfloor n/2\rfloor$ edges.

Vizing's proof of his theorem consists of constructing a $(\Delta + 1)$-edge-colouring of G edge by edge using a *recolouring procedure*. Vizing showed that, being uv the edge to be coloured, if each vertex in the closed neighbourhood of u *misses a colour* from the colour set $\mathscr{C}$ being used (i.e. no edge incident with u is coloured α for some $\alpha \in \mathscr{C}$), then we can always obtain a colour from $\mathscr{C}$ to assign to uv, possibly recolouring other edges of G. Clearly, Vizing's condition for his recolouring procedure always holds when $\mathscr{C}$ has $\Delta + 1$ colours.

In 2020, Zatesko et al. presented a novel recolouring procedure useful to construct Δ-edge-colourings. Their procedure *extends* Vizing's because in the condition to colour edge uv, each neighbour x of u is no longer required to *actually* miss a colour α, but instead x can miss α *virtually*, as the authors define. This way, if $\sum_{w\in N_G(x)}(\Delta - d_G(w)) \geq \Delta$ for every neighbour x of u that does not miss any colour from $\mathscr{C}$, then Zatesko et al. showed that we can always obtain a colour from $\mathscr{C}$ to assign to uv, possibly recolouring other edges.

In this project, we aim to investigate results from the literature that can have alternative simpler proofs using Zatesko et al.' recolouring procedure. For instance, the sufficiency of Plantholt's Theorem (remark that the necessity is immediate), whose original proof goes through 7 pages of his paper, is such a proof. As noticed by Plantholt, if G is an n-vertex non-overfull graph with a universal vertex, then either n is even (and we are done), or the complement of G has at least $(n-1)/2$ edges. In the latter case, $\sum_{w\in N_G(x)}(\Delta - d_G(w)) \geq \Delta = n - 1$ for every universal vertex x of G, so the Δ-colouring can be constructed using Zatesko et al.' recolouring procedure. As a corollary, we have an alternative proof for the characterisation of the chromatic index of split-indifference graphs by Ortiz et al. (1998).

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