On a conjecture on edge-colouring join graphs

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ABSTRACT

We bring new insights on a recent conjecture on edge-colouring. The conjecture states that if G is the disjoint join of two graphs with same order and same maximum degree such that the vertices of maximum degree of one of them induce an acyclic graph, then G is $\Delta(G)$-edge-colourable. We present a polynomial-time heuristic for $\Delta(G)$-edge-colouring such graphs. Our algorithm may fail in a specific case, and we conjecture that it is always possible to handle this case.

Keywords

Edge-colouring; Join graphs; Recolouring procedures

1. INTRODUCTION

Computing the chromatic index of an n-order graph G is an NP-hard [3] problem, but a conjecture proposed in 1984 [1] would imply, if proved, that the problem can be solved in linear time if $\Delta(G) \geqslant n/2$ [7], as all join graphs have. Linear-time algorithms for the chromatic index of join graphs are known only for some cases and the problem remains open even restricted to disjoint joins of graphs with same order and same maximum degree [5, 6, 8, 9, 10, 11, 13].

Let G be the join of two disjoint graphs G_1 and G_2 with same order and same maximum degree. In [5] the authors conjectured that $\chi'(G) = \Delta(G)$ whenever G_1 and G_2 are cographs and $\Lambda[G_1]$ is acyclic (see definitions in the sequel). The authors showed that their conjecture holds when $|V(G_2) \setminus V(\Lambda[G_2])| \geqslant |V(\Lambda[G_1])|$. This conjecture was extended for the case wherein G_1 and G_2 are not required to be cographs [13], and in the same paper proved under

$$|V(G_2) \setminus V(\Lambda[G_2])| \geqslant |\{u \in V(\Lambda[G_1]) : d_{\Lambda[G_1]}(u) > 1\}| + |\{C \text{ connected component of } \Lambda[G_1] : |V(C)| = 2\}|.$$

We provide a partial proof for the latter conjecture (and consequently for the former), not imposing $V(G_2) \setminus V(\Lambda[G_2]) \neq \emptyset$. Our proof can be viewed as a polynomial-time heuristic to obtain a $\Delta(G)$-edge-colouring. We use the term *heuristic* since there is a very specific case in which we do not know yet how to proceed. We identify this case and we conjecture that it can always be handled in polynomial time.

This paper is organised as follows. The remaining of this section provides some of the definitions used. Section 2 presents the above-mentioned proof and other accessory results. Section 3 concludes with remarks for future works.

Preliminary definitions

In this work, a *graph* is a simple graph, that is, an undirected loopless graph with no multiple edges. The *degree* of a vertex u in a graph G is denoted by $d_G(u) := |N_G(u)| = |\partial_G(u)|$, wherein $N_G(u)$ denotes the set of the neighbours of u in G and $\partial_G(u)$ denotes the set of the edges incident to u in G. Also, for any $X \subseteq V(G)$, we denote by $\partial_G(X)$ the *cut* defined by X in G, i.e. the set of the edges of G with exactly one endpoint in X. At last, we denote by $\Lambda[G]$ the *core* of G, i.e. the subgraph of G induced by all its vertices of maximum degree. Other graph-theoretical definitions follow their usual meanings and notation found in the literature.

A *t-edge-colouring* of G is a function $\varphi\colon E(G) \to \mathscr{C}$ such that $|\mathscr{C}| = t$ and adjacent edges have different images (or *colours*) assigned by φ. We say that a vertex u *miss* some colour $\alpha \in \mathscr{C}$, and that α is *missing at* u, if no edge incident to u is coloured with α. The *chromatic index* $\chi'(G)$ of G is the least t for which G is t-edge-colourable. In 1964, Vizing showed that either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ for every graph G [12], in which case G is said to be respectively *Class 1* or *Class 2*. Vizing's proof is based on a polynomial-time recolouring procedure under which a $(\Delta(G)+1)$-edge-colouring of G can be constructed edge by edge. The same procedure can be used to show that every graph with acyclic core is *Class 1*, colouring the edges in a convenient order [2].

The *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 * G_2$, is given by $V(G_1 * G_2) := V_1 \cup V_2$ and $E(G_1 * G_2) := E_1 \cup E_2 \cup \{v_1v_2 : v_1 \in V_1 \text{ and } v_2 \in V_2\}$. A *join graph* is a K_1 graph or the result of the join of two disjoint

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graphs. The K_1 is considered a join graph because, in a join operation $G_1 * G_2$, the graphs are not required to be disjoint. Nevertheless, as pointed by [13], if $G_1 * G_2 \neq K_1$, then there are disjoint graphs G'_1 and G'_2 such that $G_1 * G_2 = G'_1 * G'_2$.

2. THE HEURISTIC

We start with a proposition on general graphs. This proposition is used in the proof of Theorem 1, the main theorem which completes the presentation of our heuristic. Precede the theorem two lemmas and a conjecture, which discuss a recolouring procedure on which our heuristic is based.

Proposition 1. *If a graph G has maximum degree $\Delta > 1$ and an acyclic core with s vertices, then G has at least*

$$\max\left\{\Delta - 1, s - \left\lfloor \frac{s-2}{\Delta - 1} \right\rfloor\right\}$$

vertices of degree less than Δ.

Proof. Consider the set

$$X \coloneqq \{(u, e) : u \in V(\Lambda[G]) \text{ and } e \in \partial_G(u)\}.$$

As each $u \in V(\Lambda[G])$ appears in X exactly $d_G(u) = \Delta$ times, we have $|X| = s\Delta$. On the other hand, in view of $\bigcup_{u \in V(\Lambda[G])} \partial_G(u) = E(\Lambda[G]) \cup \partial_G(V(\Lambda[G]))$, each edge of $E(\Lambda[G])$ appears in X exactly twice, whereas each edge of $\partial_G(V(\Lambda[G]))$ appears in X exactly once, which brings $|X| = 2|E(\Lambda[G])| + |\partial_G(V(\Lambda[G]))|$. Since $\Lambda[G]$ is acyclic, $|E(\Lambda[G])| \leqslant s - 1$. Therefore, $|\partial_G(V(\Lambda[G]))| \geqslant s(\Delta - 2) + 2$.

Because $|\partial_G(u) \cap \partial_G(V(\Lambda[G]))|$ is at most s and at most $\Delta - 1$ for all $u \in V(G) \setminus V(\Lambda[G])$, and since $|\partial_G(V(\Lambda[G]))| = \sum_{u \in V(G) \setminus V(\Lambda[G])} |\partial_G(u) \cap \partial_G(V(\Lambda[G]))|$, we have $(n - s)s \geqslant s(\Delta - 2) + 2$ and $(n - s)(\Delta - 1) \geqslant s(\Delta - 2) + 2$, implying

$$n - s \geqslant \left\lceil (\Delta - 2) + \frac{2}{s} \right\rceil \geqslant \Delta - 1 \quad \text{and}$$

$$n - s \geqslant \left\lceil s - \frac{s-2}{\Delta - 1} \right\rceil = s - \left\lfloor \frac{s-2}{\Delta - 1} \right\rfloor,$$

since $n - s$ is an integer and $s > 0$. ☐

The lower bound of Proposition 1 is tight, being the diamond $(K_2 * \overline{K_2})$ an example of a graph with $\Delta - 1 = s - \lfloor (s-2)/(\Delta - 1) \rfloor = 2$ vertices not in its acyclic core.

From now on, let G be the join of two disjoint k-order graphs G_1 and G_2 with same maximum degree d. Recall that $\Delta(G) = k + d$. Let $B_G \coloneqq G - (E(G_1) \cup E(G_2))$, $G_M \coloneqq (G_1 \cup G_2) + M$ for any perfect matching M on B_G, and $M(x)$ be the vertex matched to x by M for any $x \in V(G_M)$. As observed by [8], G is *Class 1* whenever G_M is *Class 1* for some perfect matching M on B_G, since $G = G_M \cup (B_G - M)$, $\Delta(G_M) = d + 1$, and $B_G - M$ is bipartite, hence *Class 1* [4].

Our recolouring procedure is similar to Vizing's [12], for which we extend and adapt the concept of a recolouring fan. In Definition 1, in Lemmas 1 and 2, and in Conjecture 1, M is a perfect matching on B_G, $uv \in E(\Lambda[G_1])$, $\mathscr{C}$ is a set with $d+1$ colours, and $\varphi\colon E(G_M - uv) \to \mathscr{C}$ is an edge-colouring.

Definition 1. A sequence $v_0, \ldots, v_k$ of distinct neighbours of u in G_M is a *recolouring fan* for u if $v_0 = v$ and, for all $i \in \{0, \ldots, k-1\}$, either v_i *actually* misses the colour $\alpha_i \coloneqq \varphi(uv_{i+1})$, or v_i misses α_i *virtually*, that is, $i > 0$, $v_i = M(u)$, and $\varphi(wM(w)) = \alpha_i$ for some $w \in V(G_1) \setminus \{v_{i-1}\}$ which actually misses α_{i-1}. If v_k misses, actually or virtually, a colour α_k missing at u, the fan is said to be *complete*; otherwise, it is said to be *incomplete*.

The figure below illustrates a complete recolouring fan. The dashed line indicates the edge to be coloured and the dotted lines indicate the colours missing at the vertices.

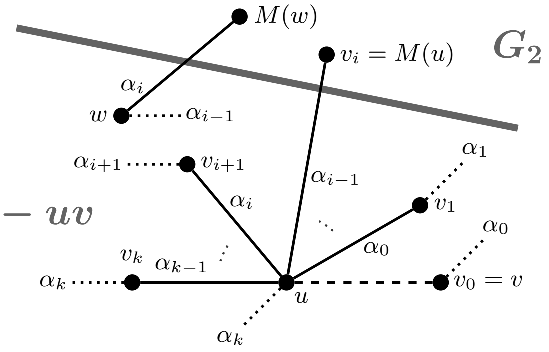

Figure 1: A complete recolouring fan

Lemma 1. *If there is a complete recolouring fan for u, then $G_{M'}$ is* Class 1 *for some perfect matching M' on B_G.*

Proof. We shall perform a procedure called *decay of the colours*, for i from k down to 0. At the beginning of each iteration, it shall be invariant that both u and v_i miss α_i (the latter possibly virtually). Note that this is true for $i = k$.

If v_i actually misses α_i, we simply assign α_i to uv_i. If $i = 0$, we are done. If $i > 0$, u now misses α_{i-1}, which is still missing (possibly virtually) at v_{i-1}, so we can continue.

If v_i misses α_i virtually, recall that $i > 0$ and $v_i = M(u)$. Then, we take $M' \coloneqq (M \setminus \{uv_i, wM(w)\}) \cup \{uM(w), wv_i\}$, colouring $uM(w)$ and uv_i with respectively α_i and α_{i-1} (see Figure 2). Now, both u and v_{i-1} actually miss α_{i-1}. ☐

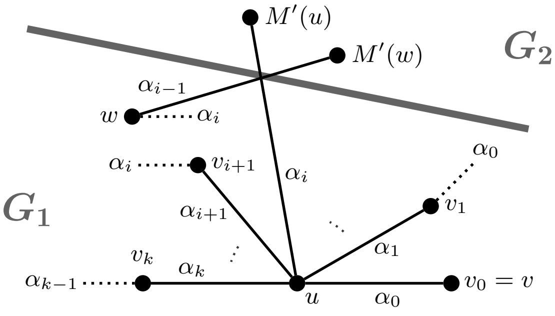

Figure 2: The result of the decay of the colours on the complete recolouring fan of Figure 1

Conjecture 1. *If $v_0, \ldots, v_k$ is a maximal, but not complete, recolouring fan for u such that $v_k = M(u)$, then there are some perfect matching $\overline{M}$ on B_G and some $(d+1)$-edge-colouring of $G_{\overline{M}}$, both obtainable in polynomial time, under which there is a complete recolouring fan for u or a maximal recolouring fan for u starting in v_0 but not ending in $\overline{M}(u)$.*

Lemma 2. *If Conjecture 1 holds for all non-complete maximal recolouring fan for u ending in $M(u)$, and if, for all $y \in N_{G_M}(u)$, either y misses a colour of $\mathscr{C}$, or $y = M(u)$ and $\varphi(uy)$ is missed by at least two vertices in $V(G_1)$, then $G_{M'}$ is* Class 1 *for some perfect matching M' on B_G.*

Proof. Let $F = v_0, \ldots, v_k$ be a maximal recolouring fan for u. If F is complete, the proof follows immediately from Lemma 1. Otherwise, as v_0 is itself a (not necessarily complete) recolouring fan for u, remark that $k \geqslant 0$. Moreover, if $M(u) = v_i$ for some $i \in \{1, \ldots, k\}$, the conditions of the statement imply that there must be some $w \in V(G_1) \backslash \{v_{i-1}\}$ which miss $\varphi(uv_i) = \alpha_{i-1}$. Ergo, the only reason why F is not complete is because every colour α missing (actually or virtually) at v_k is equal to α_j for some $j < k$.

Let $\alpha = \alpha_j$ for some $j < k$ be a colour missing (actually or virtually) at v_k, β be any colour missing at u, and e be the edge incident to v_k coloured with β. Observe that $j < k-1$, as $\alpha_{k-1} = \varphi(uv_k)$, and also that every component of the subgraph H of G_M induced by the edges coloured with α or β is a path or an even cycle. We have the following cases:

Case 1. The vertex v_k actually misses α.

Case 2. The vertex v_k misses α virtually.

In Case 1, the component of H to which e belongs is a path P, wherein v_k is one of its outer vertices. Exchanging the colours along P, we have the following sub-cases:

1. If the other outer vertex of P is u (which implies that $uv_{j+1} \in E(P)$), $v_j \notin V(P)$ and, thus, after the colour exchanging operation, both u and v_j miss α (the latter possibly virtually). Now, $F' := v_0, \ldots, v_j$ is a complete recolouring fan for u, so we are done by Lemma 1.
2. If the other outer vertex of P is v_j, then $u \notin V(P)$ and, thus, after exchanging the colours along P, both u and v_j miss β (the latter possibly virtually). As in the previous sub-case, $F' := v_0, \ldots, v_j$ is now a complete recolouring fan for u and Lemma 1 applies.
3. If the other outer vertex of P is neither u nor v_j, then, after the exchanging operation, u still misses β, v_j still misses α_j, and F is thus still a recolouring fan. But now F is complete, since now v_k misses β, so we apply Lemma 1, but in this sub-case in F instead of in F'.

In Case 2, $v_k = M(u)$ and there is some $w \in V(G_1) \setminus \{v_{k-1}\}$ which misses α_{k-1} such that $\varphi(wM(w)) = \alpha$ (see Figure 3). This is the case wherein our heuristic fails, but, if Conjecture 1 is true, we can handle this, ending up with a complete recolouring fan or going back to Case 1. ☐

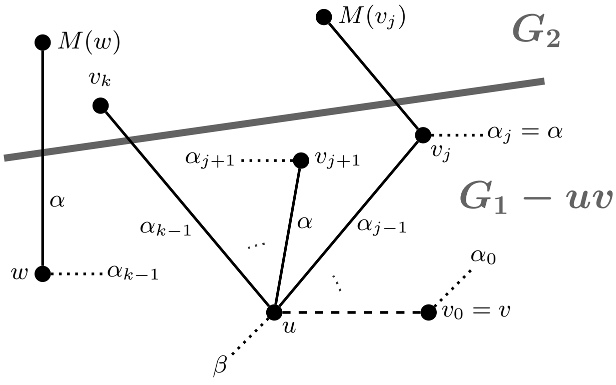

Figure 3: Case 2 in the proof of Lemma 2

Theorem 1. *Let G be the join of two disjoint k-order graphs G_1 and G_2 with same maximum degree d. If $\Lambda[G_1]$ is acyclic and Conjecture 1 is true with respect to any $uv \in E(\Lambda[G_1])$, any perfect matching M on B_G, and any non-complete maximal recolouring fan for u ending in $M(u)$, then G is* Class 1.

Proof. We assume that $d > 1$, since otherwise G_1 and G_2 are disjoint unions of cliques and we already know that G is *Class 1* by [8]. For each of the components of $\Lambda[G_1]$, which are trees, choose a vertex to be the *root* of the tree. For each $u \in V(\Lambda[G_1])$, let $h(u)$ be the *depth* of u in its tree, i.e. the number of edges in the unique path between u and the root of its tree. If $h(u) > 0$, let also $p(u)$ denote the *parent* of u, i.e. the unique neighbour of u in $\Lambda[G_1]$ with depth equal to $h(u) - 1$. Consider the non-root vertices in $V(\Lambda[G_1])$ in a non-decreasing order of depth $\sigma = u_1, \ldots, u_s$. If G_2 is not regular, take a perfect matching M on B_G such that $M(u_s) \notin V(\Lambda[G_2])$. Otherwise, take any perfect matching M on B_G. As proved in [5], G_M is *Class 1* if $\Lambda[G_1]$ is edgeless, so $G_M - E(\Lambda[G_1])$ has an edge-colouring φ using a colour set $\mathscr{C}$ with $|\mathscr{C}| = d + 1$.

Now, take the non-root vertices in $V(\Lambda[G_1])$, one at each time, following the order σ. For each u taken, we shall colour the edge $up(u)$. This shall complete the $(d+1)$-edge-colouring of G_M, possibly replacing, at each step, the current matching in the role of M with another perfect matching on B_G. However, if G_2 is not regular, the edge $u_sM(u_s)$ shall never be replaced.

In each step of our algorithm, let u be the non-root vertex of $V(\Lambda[G_1])$ taken. The only neighbour of u which may not miss a colour of $\mathscr{C}$ is $M(u)$, because for every $x \in N_{\Lambda[G_1]}(u)$, either $x = p(u)$ or $h(x) > h(u)$, so the edge ux has not been coloured yet. If $u \neq u_s$, we have the following cases to investigate, with $\alpha := \varphi(uM(u))$:

Case 1. No vertex in G_1 misses α.

Case 2. At least two vertices in $V(G_1) \setminus \{u_s\}$ miss α.

Case 3. At most one vertex in $V(G_1) \setminus \{u_s\}$ misses α.

In Case 1, no recolouring fan for u starting in $v_0 = p(u)$ will contain $M(u)$, which means that every vertex in the fan will miss a colour. So, we will be able to apply Vizing's usual recolouring procedure and thence colour $up(u)$.

In Case 2, since we have assumed Conjecture 1, we can apply Lemma 2 in order to colour $up(u)$, and do so preserving the edge $u_sM(u_s)$ in M.

In Case 3, we must recall that $\sum_{v \in V(G_1)}(d - d_{G_1}(v)) \geqslant d - 1$, from Proposition 1. As $u \neq u_s$, at least two edges of $E(\Lambda[G_1])$ have not been coloured yet, one of them being $u_sp(u_s)$. Ergo, if H is the subgraph of G_M induced only by the coloured edges, $\sum_{v \in V(G_1) \setminus \{u_s\}}((d+1) - d_H(v)) = \sum_{v \in V(G_1)}((d+1) - d_H(v)) - 1$. Furthermore,

$$\sum_{v \in V(G_1)} \big((d+1) - d_H(v)\big) \geqslant \sum_{v \in V(G_1)} (d - d_{G_1}(v)) + 4.$$

Consequently,

$$\sum_{v \in V(G_1) \setminus \{u_s\}} \big((d+1) - d_H(v)\big) \geqslant d + 2.$$

By the Pigeonhole Principle, this means that there must be a colour γ missed by at least two vertices in $V(G_1) \setminus \{u_s\}$.

Looking at the subgraph H' of G_M induced by the edges coloured with γ or α, it is not hard to verify that there is some maximal path in H' along whose edges the exchanging of the colours brings us back to one of the previous cases.

Finally, let us consider the last step, when $u = u_s$. Defining again H as the subgraph of G_M induced by the coloured edges, remark that

$$\sum_{v \in V(G_1)} ((d+1) - d_H(v)) \geqslant d+1. \qquad (1)$$

If G_2 is not d-regular, $M(u) \notin V(\Lambda[G_2])$ and we can apply the usual Vizing's recolouring procedure in order to colour $up(u)$. Assume then that G_2 is regular, which implies that no vertex in G_2 misses a colour of $\mathscr{C}$. We claim that we must have at least one colour missed by at least two vertices in G_1, so the proof can follow analogously as in the previous steps. If we assume, by the sake of contradiction, that no colour of $\mathscr{C}$ is missed by more than one vertex in G_1, we have by (1) that every colour γ of $\mathscr{C}$ is missed by exactly one vertex v_γ. If this is true, then also $v_{\gamma_1} \neq v_{\gamma_2}$ whenever $\gamma_1 \neq \gamma_2$ for all $\gamma_1, \gamma_2 \in \mathscr{C}$, because $U \cup \{u, p(u)\}$ is a set with $d+1$ vertices of degree less than $d+1$ in H for any set $U \subset V(G_1)$ with $d-1$ vertices of degree less than d in G_1. The existence of such U is guaranteed by Proposition 1. But then, creating a new vertex b in H and, for all $\gamma \in \mathscr{C}$, creating the edge bv_γ in H and colouring it with γ, H would be an odd-order *Class 1* regular graph, something impossible to happen, as one can easily verify. Therefore, the claim holds. ☐

Figure 4 illustrates the proof of Theorem 1 for the join of a diamond in the role of G_1 with a K_4 in the role of G_2, which have both maximum degree $d = 3$. The figure depicts a perfect matching M on B_G, an edge-colouring of $G_M - E(\Lambda[G_1]) = G_M - up(u)$ with a colour set $\mathscr{C} = \{1, 2, 3, 4\}$, and a complete recolouring fan v_0, v_1, v_2 for u.

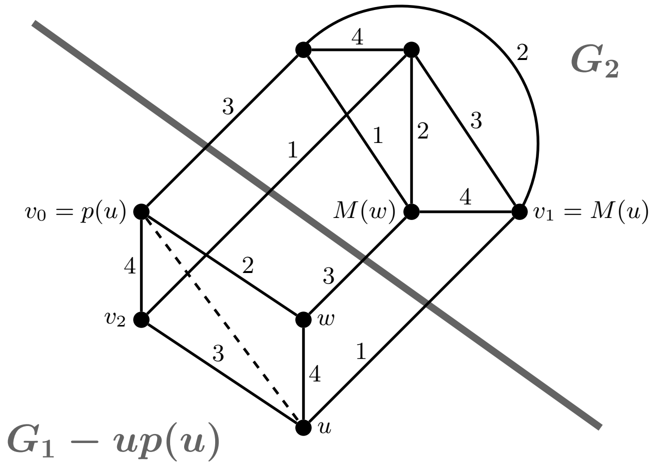

Figure 4: A perfect matching M on B_G when G_1 is a diamond and $G_2 = K_4$, a $(d+1)$-edge-colouring of $G_M - E(\Lambda[G_1])$, and a complete recolouring fan

3. FINAL REMARKS

We conclude with further discussion on Case 2 in the proof of Lemma 2. This is the case illustrated in Figure 3, which we have not been able to solve yet and for which we proposed Conjecture 1. We remark that there are some sub-cases with which we know how to deal, as what follows enlightens.

For example, let H be the subgraph of G_M induced by the edges coloured with α or β, and let X be the component of H to which $wM(w)$ belongs. We know that X can be a path or an even cycle. If u is not in X, exchanging the colours of the edges of X brings that:

1. either $v_0, \ldots, v_j$ is a complete recolouring fan, because v_j is in X and now actually misses β,
2. or $v_0, \ldots, v_k$ is a complete recolouring fan, because v_j is not in X and now v_k misses β virtually.

Either way, we know how to proceed by Lemma 1. This is sufficient to prove Conjecture 1 for the sub-case wherein u is not in X. Hence, the only remaining sub-case to prove is when u is in X. We encourage future works to investigate this sub-case and complete the proof.

4. ACKNOWLEDGEMENTS

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