On Subclasses of Circular-Arc Bigraphs

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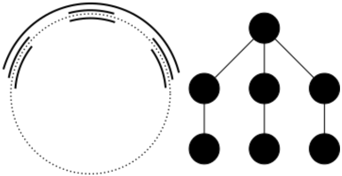

Works on circular-arc bigraphs

Lin and Szwarcfiter, 2009 *Characterizations and recognition of circular-arc graphs and subclasses: A survey* [1]
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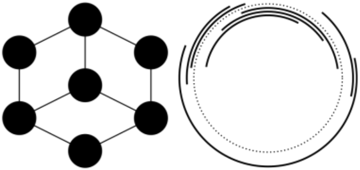

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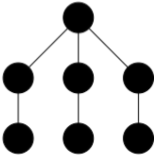

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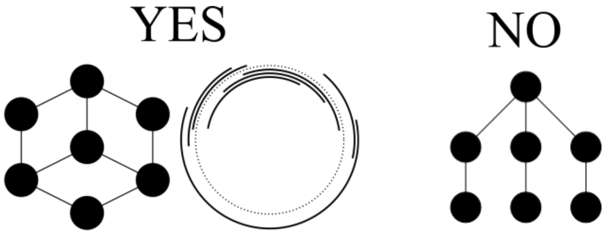

Das and Chakraborty, 2015 *New Characterizations of Proper Interval Bigraphs and Proper Circular Arc Bigraphs* [3] presents multiple characterizations of proper circular-arc bigraphs.

Helly CA bigraphs

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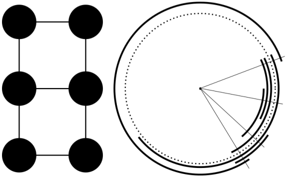

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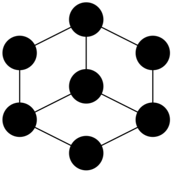

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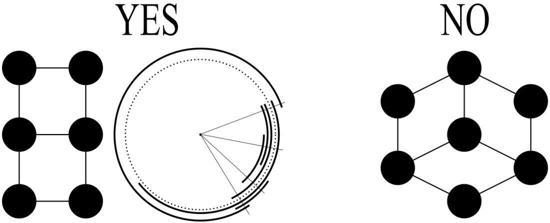

Groshaus et al., 2019 *Subclasses of Circular-Arc Bigraphs: Helly, Normal and Proper* [4]
provides a characterization and polynomial-time recognition algorithm for the class of Helly circular-arc bigraphs.

Circular convex bipartite (CCB) graphs

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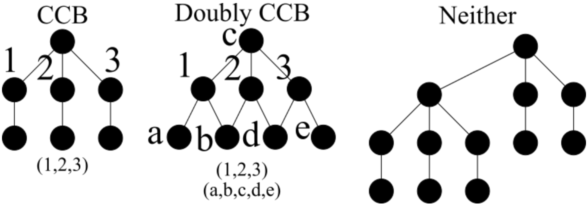

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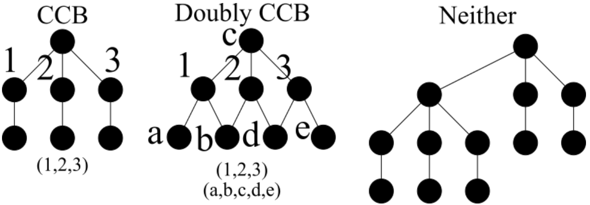

Various Works various works on CCB graphs exist, mostly focused on solving hard computational problems restricted to the class [5, 6].

Our work

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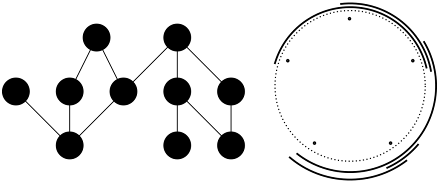

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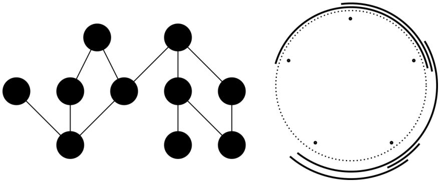

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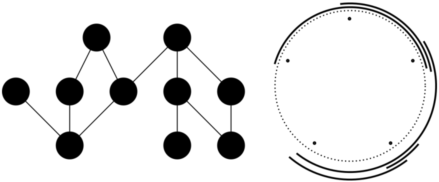

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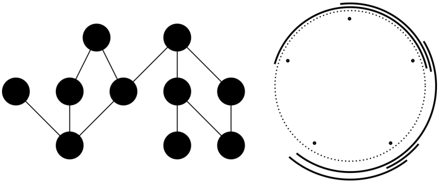

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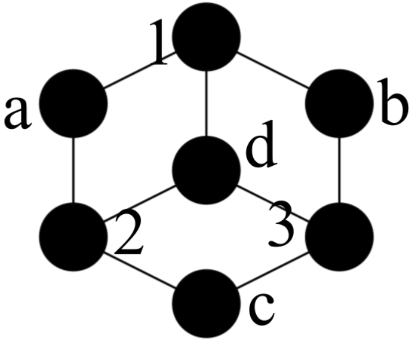

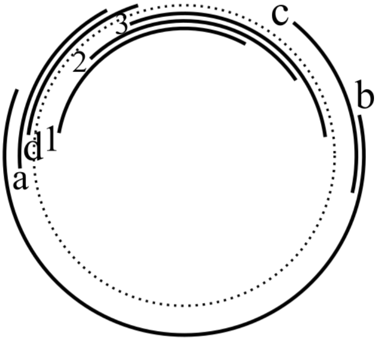

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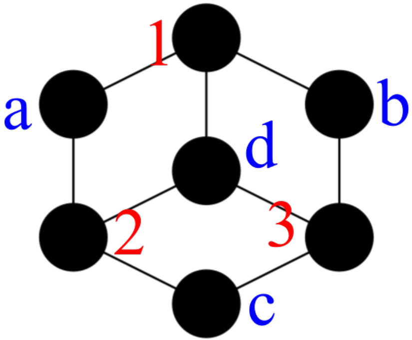

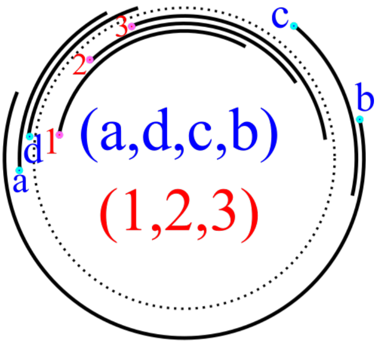

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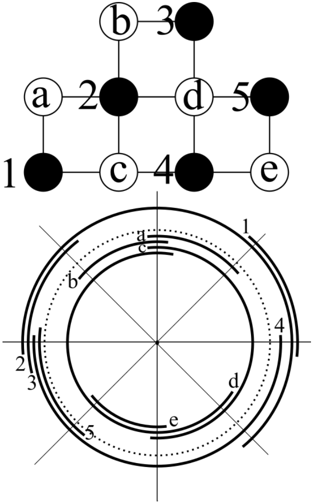

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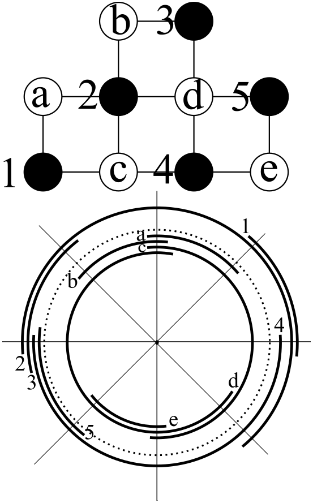

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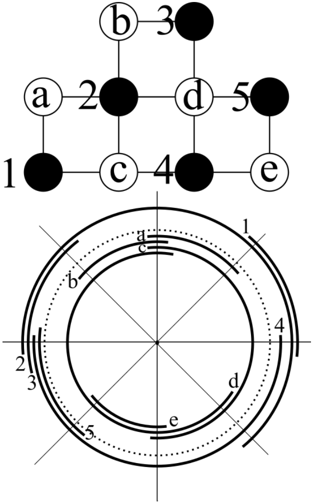

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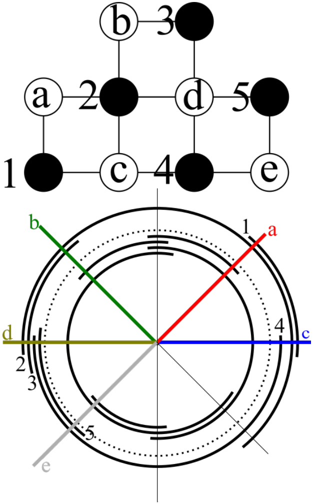

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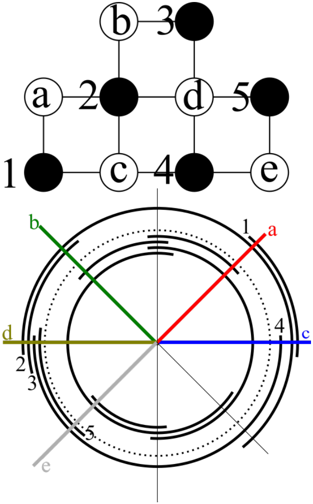

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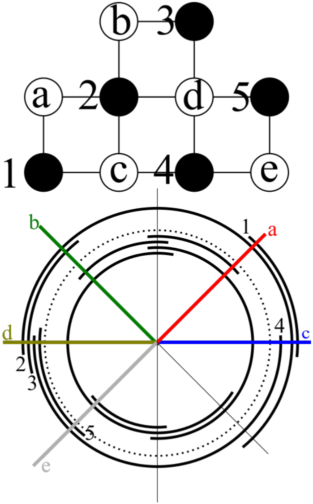

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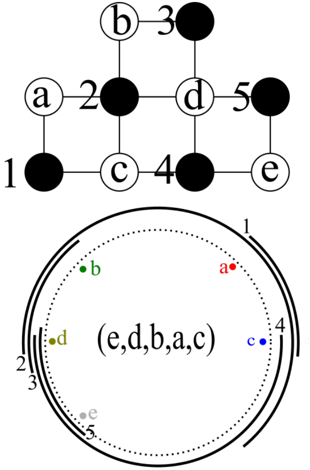

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A bipartite graph $G = (V, W, E)$ without bi-universal vertices is a proper CA bigraph if and only if V admits a CCB order such that, for every pair of elements $w_1, w_2 \in W$ satisfying $N(w_1) \subset N(w_2)$, the intervals of the CCB order corresponding to the vertices in $N(w_1)$ and $N(w_2)$ either begin or end in the same vertex.

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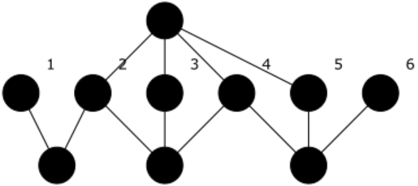

In the above graph, with order $(1, 2, 3, 4, 5, 6)$, every pair of vertices of the opposing partite set whose neighborhoods are comparable are such that their interval of neighbors either begins or ends in the same point.

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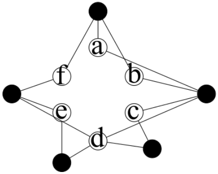

(a,b,c,d,e,f)

a	b	c	d	e	f
1	1	1	1	0	0
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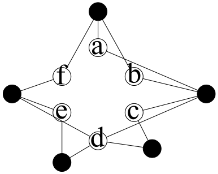

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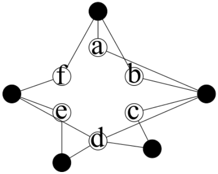

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Proof idea: this proof idea will be very rough, as the full proof would pretty much require a paper of its own.

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- $C = \{c_1, c_2, ..., c_k\}$.
- $V = \{v_1, v_2, ..., v_k\}$.
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Let the neighborhoods of the vertices in $V(G)$ be the following, for all $1 \leq i \leq k$. Consider cyclic summation (e.g. $1 - 1 = k, k + 1 = 1$) for vertex and set indices:

- $N(c_i) = \{c_{i-1}, c_{i+1}, v_i\} \cup W_i \cup X_{i-1}$.
- $N(v_i) = \{c_i\}$.
- $N(w_{i,j}) = \{c_i\} \cup \{x_{i,l} \in X_i | l \leq j\}$, for all $1 \leq j \leq n_i$.
- $N(x_{i,j}) = \{c_{i+1}\} \cup \{w_{i,l} \in W_i | l \geq j\}$, for all $1 \leq j \leq n_i$.

Results

Example of such a graph, with $k = 8$, $n_1 = 3$, $n_2, n_4, n_7, n_8 = 1, n_3, n_5, n_6 = 2$.

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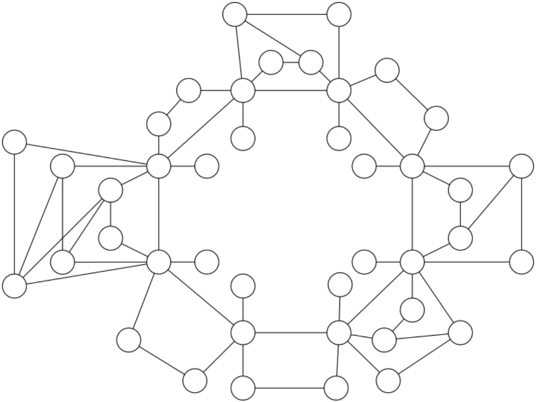

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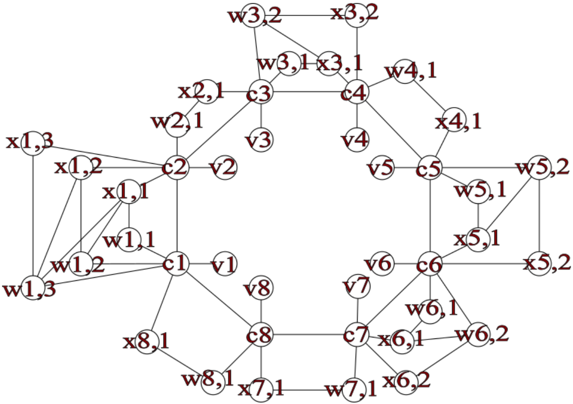
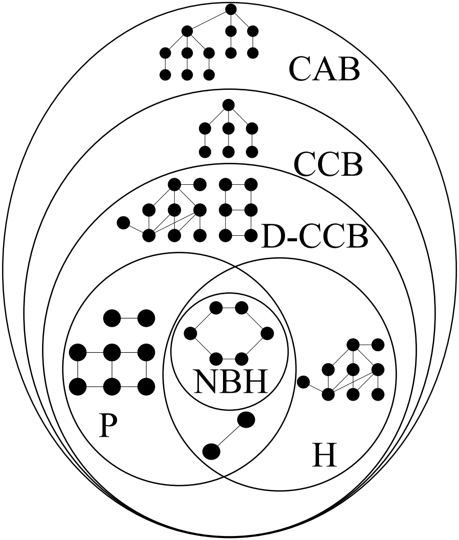


Results

We then conclude the proof by demonstrating that every graph $G = (V, W, E)$ constructed in that manner admits a CCB order such that, for every pair of elements $w_1, w_2 \in W$ satisfying $N(w_1) \subset N(w_2)$, the intervals of the CCB order corresponding to the vertices in $N(w_1)$ and $N(w_2)$ either begin or end in the same vertex.

Results

Venn diagram of the presented classes.



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Conclusion

- We provided a characterization of CCB graphs in terms of arc-point models.
- We showed that Helly CA bigraphs and proper CA bigraphs are a proper subclass of doubly CCB graphs.
- We provided a characterization for proper CA bigraphs without bi-universal vertices in terms of CCB orders.
- We showed that non-bichordal Helly CA bigraphs are proper CA bigraphs.

Thank you

Thank you all so much for watching!

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