$\mathcal{P} = \mathcal{NP}$ or 5-Snarks Exist

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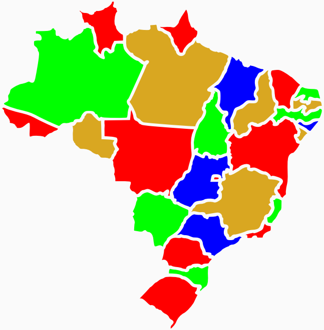
20 september 2019

Introduction

Origins

The Four Colour Conjecture (Francis Guthrie, 1852)

Every map is 4-colourable



Origins

The Four Colour Theorem (Appel et al., 1977)

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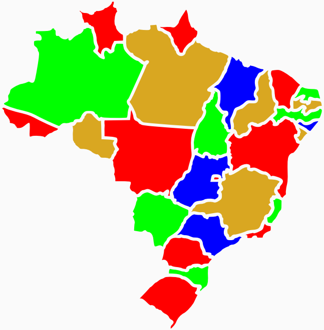

K. Appel and W. Haken (1977). Every planar map is four colorable. Part I: Discharging. *Illinois J. Math.* 21 (3), pp. 429–490.

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The Four Colour Theorem is equivalent to the statement that no ***snark is planar***

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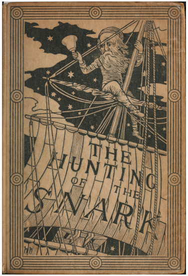

Lewis Carrol, 1876

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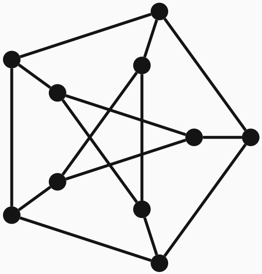
Previous results



Petersen Graph, 1898

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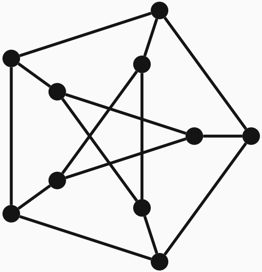

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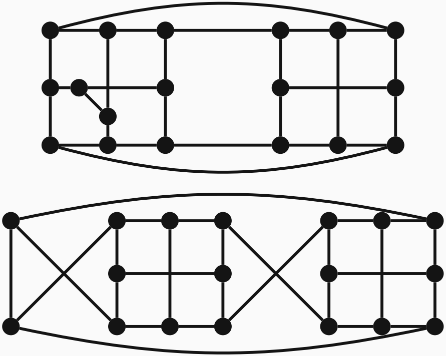

Blanuša Snarks, 1946

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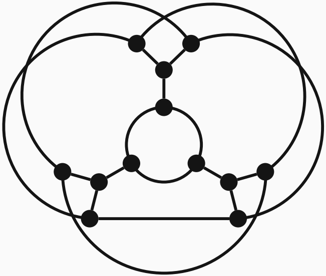

Flower Snark J_3

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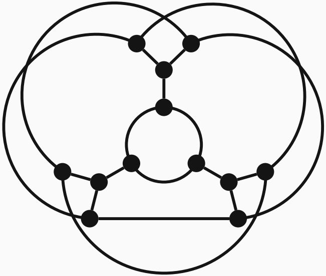

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$$2^{(n-84)/18}$$

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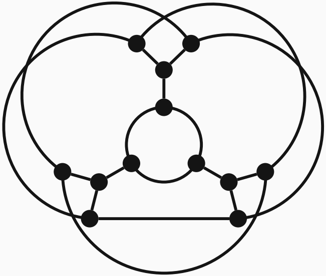

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- still ***rares*** [Robinson and Wormald 1994]

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A ***5-snark*** is a 5-regular 4-edge-connected non-5-edge-colourable graph

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Not only 5-snarks ***must exist***

Corollary

There cannot be a ***polynomial*** $p(n)$ which bounds above the number of 5-snarks of order n, unless $\mathcal{P} = \mathcal{NP}$

Related open graph problems

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- Overfull Conjecture

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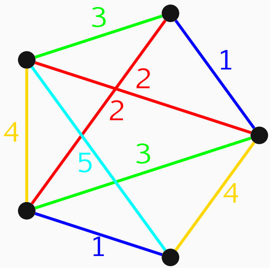
- 1-Factorisation Conjecture

Overfull Conjecture

A graph G is said to be ***overfull*** if $|E(G)| > \Delta\lfloor n/2 \rfloor$

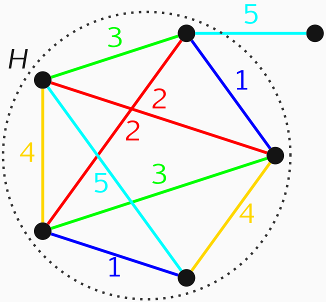


$$|E(G)| = 9$$

$$\Delta\left\lfloor \frac{n}{2} \right\rfloor = 4 \cdot 2 = 8$$

Overfull Conjecture

A graph G is said to be ***subgraph-overfull*** (shortly, *SO*) if it has an overfull Δ-subgraph H



$$|E(G)| = 10$$

$$\Delta \left\lfloor \frac{n}{2} \right\rfloor = 4 \cdot 3 = 12$$

Overfull Conjecture
(Chetwynd and Hilton, 1984, 1986; Hilton and Johnson, 1987)

If $\Delta > n/3$, then G is non-Δ-edge-colourable **if and only if** it is *SO*.

A. G. Chetwynd and A. J. W. Hilton (1984). The chromatic index of graphs of even order with many edges. *J. Graph Theory* 8, pp. 463–470.

A. G. Chetwynd and A. J. W. Hilton (1986). Star multigraphs with three vertices of maximum degree. *Math. Proc. Cambridge Philos. Soc.* 100, pp. 303–317.

A. J. W. Hilton and P. D. Johnson (1987). Graphs which are vertex-critical with respect to the edge-chromatic number. *Math. Proc. Cambridge Philos. Soc.* 102, pp. 103–112.

Proofs

EDGE-COLOURING:

Instance: a graph G;

Question: is G a Δ-edge-colourable graph?

I. Holyer (1981). The $\mathcal{NP}$-completeness of edge-colouring. *SIAM J. Comput.* 10.4, pp. 718–720.

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This problem is $\mathcal{NP}$-complete[Holyer 1981] ***even*** restricted to d-regular graphs for any constant $d \geqslant 3$[Leven and Galil 1983]

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For ***any*** constant $d \geqslant 3$

EDGE-COLOURING(d-regular, $(d-1)$-edge-connected):

Instance: a d-regular $(d-1)$-edge-connected graph G;

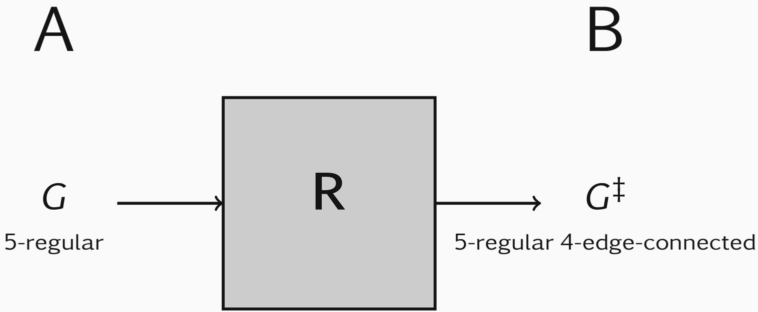
Question: is G a d-edge-colourable graph?

Our results

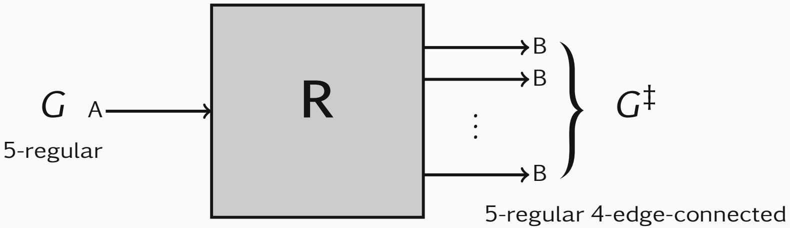
Theorem

EDGE-COLOURING(5-regular, 4-edge-connected) is $\mathcal{NP}$-complete.

Karp reduction



Turing reduction

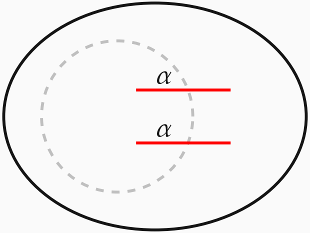


Parity Lemma

If G is d-regular d-edge-colourable graph and $F \subseteq E(G)$ is a cut in G, then, for any d-edge-colouging of G with colour set $\{1, \ldots, d\}$,

$$f_1 \equiv f_2 \equiv \cdots \equiv f_d \pmod{2},$$

being f_α the number of edges in F coloured $\alpha \in \{1, \ldots, d\}$.



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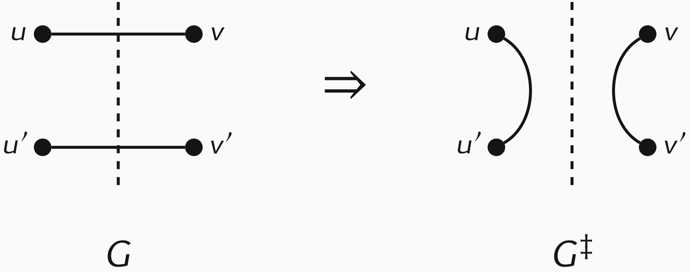

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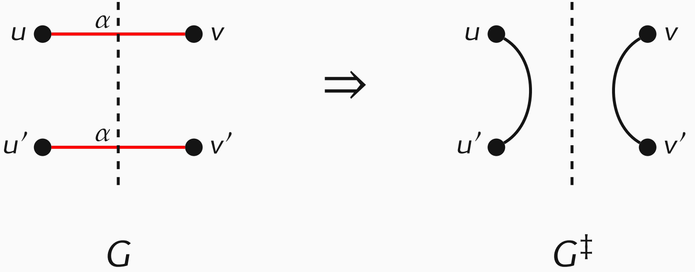

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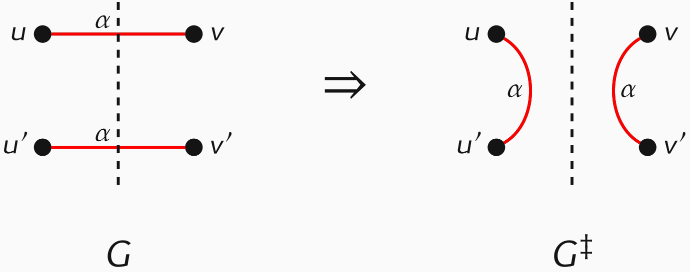

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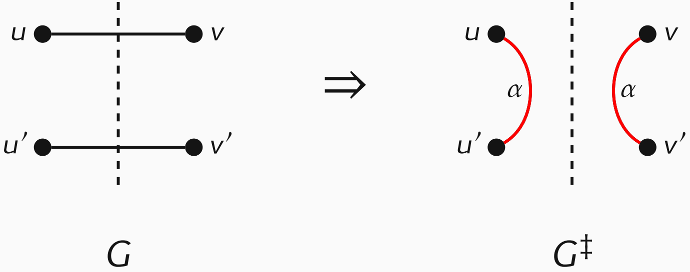

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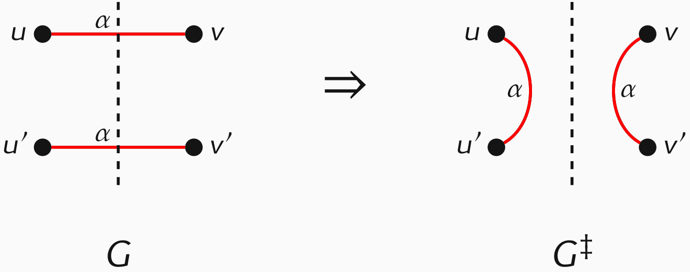

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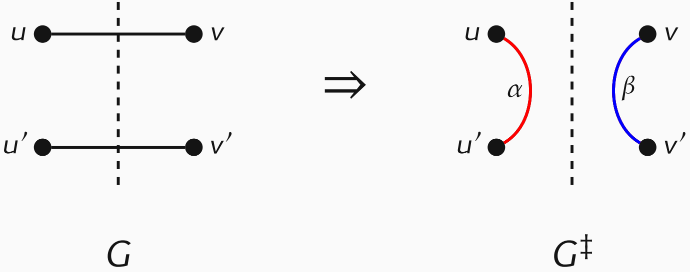

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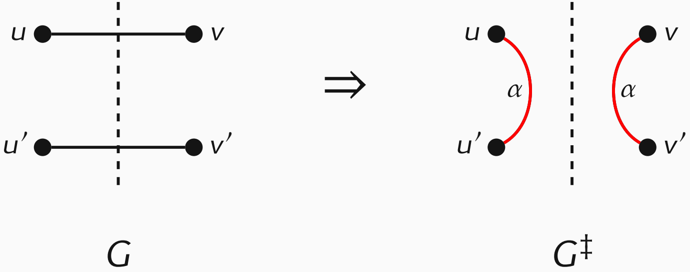

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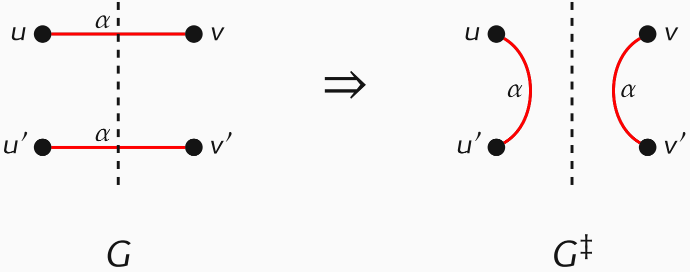

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A language L is said to be ***sparse*** if there is a polynomial $p(n)$ such that the number of words of length n belonging to L is bounded above by $p(n)$ for any n

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Fortune–Mahaney's Theorem

If any sparse language is NP-complete or coNP-complete, then P = NP

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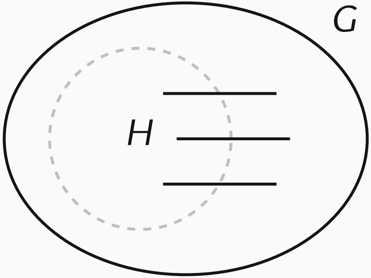

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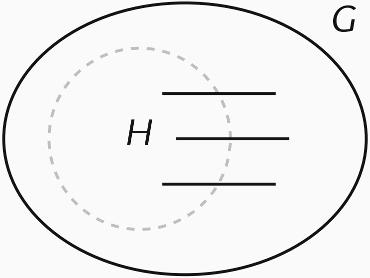

$$\sum_{u \in V(H)} (\Delta(H) - d_H(u)) \leqslant \Delta - 2$$

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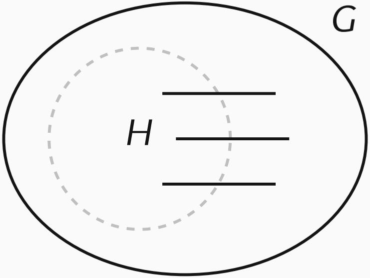

$$\sum_{u \in V(H)} (\Delta(H) - d_H(u)) \leqslant 3$$

Further investigation

Questions:

- Do 5-snarks contain 3-snarks as subgraphs?

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- Do 5-snarks contain 3-snarks as subgraphs?
- Do they allow infinite family constructions?

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